THE INTEGRALS IN GRADHTEYN AND RYZHIK. PART 5: SOME TRIGONOMETRIC INTEGRALS

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Abstract. We present evaluations and provide proofs of definite integrals involving the function $x^p \cos^n x$. These formulae are generalizations of 3.761.11 and 3.822.1, among others, in the classical table of integrals by I. S. Gradshteyn and I. M. Ryzhik.

1. Introduction

The table of integrals [4] contains a large variety of evaluations of the type

$$I = \int_a^b A(x) R(\sin x, \cos x)\, dx \tag{1.1}$$

where A is an algebraic function, R is rational and $-\infty \leq a < b \leq \infty$. We present a systematic discussion of two families of integrals of this type. This paper is part of a general program started in [9, 10, 11, 12] intended to provide proofs and context to the formulas in [4].

The first class considered here corresponds to the complete integrals

$$c(n,p) := \int_0^{\pi/2} x^p \cos^n x\, dx, \tag{1.2}$$

and

$$s(n,p) := \int_0^{\pi/2} x^p \sin^n x\, dx, \tag{1.3}$$

where $n,\ p \in \mathbb{N}$. In section 2 we present closed-form expressions for these integrals. These expressions involve the sums

$$\sum_{1 \leq k_1 \leq k_2 \leq \cdots \leq k_j \leq n} \frac{1}{k_1^2 k_2^2 \cdots k_j^2}, \tag{1.4}$$

that are closely related to the multiple zeta values

$$\zeta(i_1, i_2, \ldots, i_s) = \sum_{0 < k_1 < k_2 < \cdots < k_s} \frac{1}{k_1^{i_1} k_2^{i_2} \cdots k_s^{i_s}}. \tag{1.5}$$

The reader will find in Section 3.4 of [3] an introduction to these sums.

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In general, one does not expect such elementary evaluations to extend to $p \notin \mathbb{N}$. For example, the change of variables $x = \pi t^2/2$ produces

$$\int_0^{\pi/2} x^{-1/2} \cos x \, dx = \sqrt{2\pi} \int_0^1 \cos\left(\frac{\pi t^2}{2}\right) dt. \tag{1.6}$$

The latter integral is evaluated in terms of the *cosine Fresnel* function

$$\text{FresnelC}[x] := \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt, \tag{1.7}$$

which indeed is not an elementary function.

The second class considered here presents generalizations of the formula 3.822.1 in [4] stated as

$$\int_0^\infty \frac{\cos^{2n+1} x}{\sqrt{x}} \, dx = \frac{1}{2^{2n}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^n \binom{2n+1}{n+k+1} \frac{1}{\sqrt{2k+1}}, \quad n \in \mathbb{N}. \tag{1.8}$$

The integral in (1.8) can be transformed via $t = x^2$ to provide the evaluation of

$$\int_0^\infty \cos^{2n+1} t^2 \, dt, \tag{1.9}$$

that is given as the case $p = 2$ in Theorem 3.2.

Section 3 contains analytic expressions for the generalizations

$$C_n(p, b) := \int_0^\infty x^{-p} \cos^{2n+1}(x+b) \, dx, \tag{1.10}$$

and

$$S_n(p, b) := \int_0^\infty x^{-p} \sin^{2n+1}(x+b) \, dx. \tag{1.11}$$

The last section also contains some evaluations obtained by differentiation with respect to parameters. An illustrative example is

$$\int_0^\infty \int_0^\infty \frac{\log x \, \log y}{\sqrt{xy}} \cos(x+y) \, dx \, dy = (\gamma + 2\log 2)\pi^2, \tag{1.12}$$

that is equivalent to

$$\int_0^\infty \int_0^\infty \log x \, \log y \, \cos(x^2+y^2) \, dx \, dy = \frac{1}{16}(\gamma + 2\log 2)\pi^2. \tag{1.13}$$

A generalization of this evaluation appears as Example 3.3.

The method described in the present work gives impetus to a class of integrals that are closely related to the particular integral computations addressed in this paper.

2. The first example

In this section we present the evaluation in closed-form of the definite integrals

$$c(n, p) := \int_0^{\pi/2} x^p \cos^n x \, dx. \tag{2.1}$$

A special case of this appears as 3.822.1 in [4].

The first step towards the evaluation of $c(n, p)$ is to produce a recurrence.

Theorem 2.1. The integral $c(n, p)$ satisfies the recurrence

$$c(n, p) = \frac{n-1}{n} c(n-2, p) - \frac{p(p-1)}{n^2} c(n, p-2), \tag{2.2}$$

for $n \geq 2$, $p \geq 2$.

Proof. The identity $\cos^2 x = 1 - \sin^2 x$ yields

$$c(n,p) \quad = \quad c(n-2,p) - I(n,p) \tag{2.3}$$

where

$$I(n,p) \quad := \quad \int_0^{\pi/2} x^p \cos^{n-2} x \, \sin^2 x \, dx.$$

Now

$$\begin{aligned} I(n,p) &= \int_0^{\pi/2} x^p \sin x \times \frac{d}{dx}\left(-\frac{1}{n-1} \cos^{n-1} x \right) dx \\ &= \frac{1}{2n-1} \int_0^{\pi/2} \left(x^p \cos x + p x^{p-1} \sin x \right) \cos^{n-1} x \, dx \\ &= \frac{c(n,p)}{n-1} + \frac{p}{n-1} \int_0^{\pi/2} x^{p-1} \sin x \cos^{n-1} x \; dx. \end{aligned}$$

Moreover

$$\begin{aligned} \int_0^{\pi/2} x^{p-1} \sin x \cos^{n-1} x \; dx &= \int_0^{\pi/2} x^{p-1} \frac{d}{dx}\left(-\frac{1}{n} \cos^n x \right) dx \\ &= \frac{p-1}{n} c(n, p-2). \end{aligned}$$

□

Strategy: According to (2.2), the integral $c(n,p)$ can be evaluated in terms of the initial values given in the table. The indices m and q have the same parity as n and p respectively and range over $0 \le m \le n$ and $0 \le q \le p$.

n modulo 2	p modulo 2	initial conditions
0	0	$c(m,0)$ $c(0,q)$
1	0	$c(m,0)$ $c(1,q)$
0	1	$c(m,1)$ $c(0,q)$
1	1	$c(m,1)$ $c(1,q)$

We now evaluate the initial conditions $c(n,0)$, $c(n,1)$, $c(0,p)$ and $c(1,p)$.

The expression for $c(0,p)$.

The computation of the identity

$$c(0,p) = \frac{1}{p+1} \left(\frac{\pi}{2} \right)^{p+1} \tag{2.4}$$

is immediate.

The expression for $c(n,0)$.

This is classical. The result appears as 3.621.3 and 3.621.4 in [4].

Theorem 2.2. (Wallis' formula and companion). Let $n \in \mathbb{N}_0$. Then

$$c(2n,0) = \frac{\pi}{2^{2n+1}} \binom{2n}{n}, \tag{2.5}$$

and

$$c(2n+1,0) = \frac{2^{2n}}{(2n+1)\binom{2n}{n}}. \tag{2.6}$$

The shortest proof of Theorem 2.2 employs the representation

$$c(n,0) = \int_0^{\pi/2} \cos^n x \, dx = 2^{n-1} B\left(\frac{n+1}{2}, \frac{n+1}{2} \right), \tag{2.7}$$

that appears as 3.621.1 in [4]. Here B is the *Euler's beta function* defined by the integral

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, dt. \tag{2.8}$$

The expression (2.7) follows from the change of variables $t = \cos u$. To express (2.5) and (2.6), in terms of the beta function, employ the standard relation

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}, \tag{2.9}$$

and the special values

$$\Gamma(n) = (n-1)! \text{ and } \Gamma(n+\tfrac{1}{2}) = \frac{\sqrt{\pi}\,(2n)!}{2^{2n}\,n!} \tag{2.10}$$

that are valid for $n \in \mathbb{N}$.

The identity in Theorem 2.2, in the case n is even, that is,

$$c(2n,0) = \int_0^{\pi/2} \cos^{2n}\theta \, d\theta = \frac{\pi}{2^{2n+1}} \binom{2n}{n}, \tag{2.11}$$

is Wallis's formula and sometimes found in calculus books (see e.g. [6], page 492). To prove it, first write $\cos^2\theta = 1 - \sin^2\theta$ and use integration by parts to obtain the recursion

$$c(2n,0) \quad = \quad \frac{2n-1}{2n} c(2n-2,0). \tag{2.12}$$

Then verify that the right side of (2.11) satisfies the same recurrence together with the initial value $\pi/2$ for $n = 0$.

We now present a new proof of Wallis's formula (2.11) in the context of rational integrals. Extensions of the ideas in this proof have produced *rational Landen transformations*. The reader will find in [1, 2, 5, 7, 8] details on these transformations.

Start with

$$c(2n,0) \quad = \quad \int_0^{\pi/2} \cos^{2n}\theta \, d\theta = \int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2} \right)^n d\theta.$$

Now introduce $\psi = 2\theta$ and expand and simplify the result by observing that the odd powers of cosine integrate to zero. The inductive proof of (2.11) requires

$$c(2n,0) \quad = \quad 2^{-n} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} c(2i,0). \tag{2.13}$$

Note that $c(2n,0)$ is uniquely determined by (2.13) along with the initial value $c(0,0) = \pi/2$. Thus (2.11) now follows from the identity

$$f(n) := \sum_{i=0}^{\lfloor n/2 \rfloor} 2^{-2i} \binom{n}{2i} \binom{2i}{i} \quad = \quad 2^{-n} \binom{2n}{n}. \tag{2.14}$$

We now provide a mechanical proof of (2.14) using the theory developed by Wilf and Zeilberger, which is explained in [13, 14]; the sum in (2.14) is the example used in [14] (page 113) to illustrate their method. The command

$$ct(\text{binomial}(n, 2i)\,\text{binomial}(2i, i)2^{-2i}, 1, i, n, N)$$

produces

$$f(n+1) \quad = \quad \frac{2n+1}{n+1}\, f(n), \tag{2.15}$$

and one checks that $2^{-n}\binom{2n}{n}$ satisfies this recursion. Note that (2.12) and (2.15) are equivalent under

$$c(2n, 0) \quad = \quad \frac{\pi}{2^{n+1}} f(n).$$

The proof is complete.

Closed form expression for $c(1, p)$.

We now consider the evaluation of

$$c(1, p) := \int_0^{\pi/2} x^p \cos x \, dx. \tag{2.16}$$

The following evaluation appears as 3.761.11 in [4].

Theorem 2.3. Let $p \in \mathbb{N}$ and $\delta_{\text{odd},p}$ be Kronecker's delta function at the odd integers. Then

$$c(1, p) = \sum_{k=0}^{\xi_p} (-1)^k \frac{p!}{(p-2k)!} \left(\frac{\pi}{2}\right)^{p-2k} - (-1)^{\xi_p} \delta_{\text{odd},p}\, p! \tag{2.17}$$

where $\xi_p = \lfloor \frac{p}{2} \rfloor$.

Proof. Both sides of the equation (2.17) satisfy the initial value problem

$$u_p - p(p-1)u_{p-2} = \left(\frac{\pi}{2}\right)^p \text{ and } u_0 = 1,\ u_1 = \frac{\pi - 2}{2}. \tag{2.18}$$

Actually the recurrence (2.18) is obtained using integration by parts in (2.16). Iterating this recurrence yields the right hand side of (2.17). $\square$

Note 2.4. The result in Theorem 2.3 can be expressed in terms of the Taylor polynomial for $\cos x$:

$$f_p(x) = (-1)^{\xi_p} p! \left(-1 + \sum_{k=0}^{\xi_p} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \right). \tag{2.19}$$

The formula (2.17) can be restated

$$c(1, p) = \begin{cases} f_p(\pi/2), & \text{for } p \text{ odd}, \\ f_p'(\pi/2), & \text{for } p \text{ even}. \end{cases} \tag{2.20}$$

Closed form expression for $c(n, 1)$: in fact, this would be the last initial condition we require to execute the startegy outlined at the beginning of this section.

Theorem 2.5. The integral $c(n, 1)$ satisfies the recurrence

$$c(n, 1) \quad = \quad \frac{n-1}{n} c(n-2, 1) - \frac{1}{n^2}. \tag{2.21}$$

Proof. The identity $\cos^2 x = 1 - \sin^2 x$ yields

$$c(n,1) = c(n-2,1) - J, \tag{2.22}$$

where

$$J = \int_0^{\pi/2} x \sin^2 x \, \cos^{n-2} x \, dx. \tag{2.23}$$

Integration by parts leads to

$$J = \frac{1}{n-1} \int_0^{\pi/2} (\sin x + x \cos x) \cos^{n-1} x \, dx. \tag{2.24}$$

This produces (2.21). □

The solution of (2.21) yields a closed-form formula for $c(n,1)$.

Theorem 2.6. The integral $c(n,1)$ is given according to the parity of n, by

$$c(2n,1) = \frac{\binom{2n}{n}}{2^{2n+2}} \left(\frac{\pi^2}{2} - \sum_{k=1}^{n} \frac{2^{2k}}{k^2 \binom{2k}{k}} \right), \tag{2.25}$$

for even indices. For odd indices, we have

$$c(2n+1,1) = \frac{2^{2n}}{(2n+1)\binom{2n}{n}} \left(\frac{\pi}{2} - \sum_{k=0}^{n} \frac{\binom{2k}{k}}{2^{2k}\,(2k+1)} \right). \tag{2.26}$$

To establish this result we solve a more general recurrence than (2.21).

Lemma 2.7. Let a_n, b_n and r_n be sequences with $a_n, \, b_n \neq 0$. Assume that z_n satisfies

$$a_n z_n = b_n z_{n-1} + r_n, \; n \geq 1 \tag{2.27}$$

with initial condition z_0. Then

$$z_n = \frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n} \left(z_0 + \sum_{k=1}^{n} \frac{a_1 a_2 \cdots a_{k-1}}{b_1 b_2 \cdots b_k} r_k \right). \tag{2.28}$$

Proof. Introduce the integrating factor d_n with the property that $d_n b_n = d_{n-1} a_{n-1}$. The recurrence (2.27) becomes

$$D_n - D_{n-1} = d_n r_n, \tag{2.29}$$

where $D_n = d_n a_n z_n$. Therefore, by telescoping,

$$D_n = D_0 + \sum_{k=1}^{n} d_k r_k, \tag{2.30}$$

with $D_0 = d_0 a_0 z_0$. To find the integrating factor, observe that

$$\frac{d_n}{d_{n-1}} = \frac{a_{n-1}}{b_n}. \tag{2.31}$$

Thus

$$d_n = \frac{a_0 a_1 \cdots a_{n-1}}{b_1 b_2 \cdots b_n} d_0. \tag{2.32}$$

Replacing in (2.30) yields (2.28). □

Corollary 2.8. Let $n \in \mathbb{N}$ and assume that z_n satisfies

$$2nz_n = (2n-1)z_{n-1} + r_n,\ n \geq 1, \tag{2.33}$$

with the initial condition z_0. Let $\lambda_j = 2^{2j}\binom{2j}{j}^{-1}$, then

$$z_n = \frac{1}{\lambda_n}\left(z_0 + \sum_{k=1}^{n} \frac{\lambda_k r_k}{2k}\right). \tag{2.34}$$

Proof. Use $a_n = 2n$ and $b_n = 2n-1$ in Lemma 2.7. □

We now apply Lemma 2.7 on the recurrence (2.21), repeated here for convenience to the reader,

$$c(n,1) = \frac{n-1}{n}c(n-2,1) - \frac{1}{n^2}.$$

Observe that this recurrence splits naturally into even and odd branches. The value of $c(2n,1)$ is determined completely by $c(0,1)$, and $c(2n+1,1)$ by $c(1,1)$. Hence, there is no computational interaction between $c(2n,1)$ and $c(2n+1,1)$. Let $x_n = c(2n,1)$ so that x_n satisfies

$$2nx_n = (2n-1)x_{n-1} - \frac{1}{4n}, \tag{2.35}$$

with the initial condition

$$x_0 = c(0,1) = \frac{\pi^2}{8}. \tag{2.36}$$

Similarly, $y_n = c(2n+1,1)$, the odd component of $c(n,1)$, satisfies

$$(2n+1)y_n = 2ny_{n-1} - \frac{1}{2n+1} \tag{2.37}$$

and the initial condition

$$y_0 = c(1,1) = \frac{\pi}{2} - 1. \tag{2.38}$$

The expressions for z_n in Lemma 2.7 yield the formulas for $c(2n,1)$ and also $c(2n+1,1)$ in Theorem 2.6. The proof is complete.

Note 2.9. The finite sums in (2.25) and (2.26) do not have closed-form, but it is a classical result that, in the limit,

$$\sum_{k=1}^{\infty} \frac{2^{2k}}{k^2\binom{2k}{k}} = \frac{\pi^2}{2} \tag{2.39}$$

and

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}(2k+1)} = \frac{\pi}{2}. \tag{2.40}$$

Note 2.10. Formula 3.821.3 in [4] gives formulas equivalent to (2.25) and (2.26), respectively.

Finally, we conclude this section by presenting ithe sought for closed form expression for the integral $c(n,p)$, for arbitrary n, $p \in \mathbb{N}$. The recurrence (2.2), in the case of even indices n, becomes

$$2nX_n(p) = (2n-1)X_{n-1}(p) - \frac{p(p-1)}{2n}X_n(p-2) \tag{2.41}$$

where $X_n(p) = c(2n,p)$. The initial value

$$X_0(p) = \frac{1}{(p+1)2^{p+1}}\pi^{p+1} \tag{2.42}$$

given in (2.4) and the recurrence (2.41) show the existence of rational numbers $a_{n,p,p+1-2j}$ such that

$$X_n(p) = \sum_{j=0}^{\xi_p} a_{n,p,p+1-2j}\pi^{p+1-2j}, \tag{2.43}$$

with $\xi_p = \lfloor \frac{p}{2} \rfloor$. The recurrence (2.41) is now expanded as

$$\begin{aligned} 2n\sum_{j=0}^{\xi_p} a_{n,p,p+1-2j}\pi^{p+1-2j} &= (2n-1)\sum_{j=0}^{\xi_p} a_{n-1,p,p+1-2j}\pi^{p+1-2j} \\ &- \frac{p(p-1)}{2n}\sum_{j=0}^{\xi_{p-1}} a_{n,p-2,p-1-2j}\pi^{p-1-2j}. \end{aligned} \tag{2.44}$$

The fact that the coefficients $a_{n,p,j} \in \mathbb{Q}$ allows us to match the corresponding powers of π in (2.44). The highest order term is π^{p+1}. Only two of the sums contain this power, therefore

$$2na_{n,p,p+1} = (2n-1)a_{n-1,p,p+1}. \tag{2.45}$$

The initial condition

$$a_{0,p,p+1} = \frac{1}{(p+1)2^{p+1}} \tag{2.46}$$

comes from (2.42). The solution to the initial value problem (2.45, 2.46) is then found using Corollary 2.8 (here $r_n = 0$), namely that

$$a_{n,p,p+1} = \frac{\binom{2n}{n}}{(p+1)2^{2n+p+1}}. \tag{2.47}$$

The coefficient of the next highest power π^{p-1}, in (2.44), yields the recurrence

$$2na_{n,p,p-1} = (2n-1)a_{n-1,p,p-1} - \frac{p(p-1)}{2n}a_{n,p-2,p-1}. \tag{2.48}$$

Observe that the last term in this relation is given by (2.47). Moreover, (2.42) shows that $a_{0,p,p-1} = 0$. The solution to (2.48), following Corollary 2.8, is

$$a_{n,p,p-1} = -\frac{p\binom{2n}{n}}{2^{2n+p+1}}\sum_{k_1=1}^{n}\frac{1}{k_1^2}. \tag{2.49}$$

The next power of π in (2.44) produces

$$2na_{n,p,p-3} = (2n-1)a_{n-1,p,p-3} + \frac{p(p-1)(p-2)}{n2^{2n+p}}\binom{2n}{n}\sum_{k_1=1}^{n}\frac{1}{k_1^2}, \tag{2.50}$$

with $a_{0,p,p-3} = 0$. One more use of Corollary 2.8 yields

$$a_{n,p,p-3} = \frac{\binom{2n}{n}\,p!}{2^{2n+p+1}\,(p-3)!}\sum_{k_2=1}^{n}\sum_{k_1=1}^{k_2}\frac{1}{k_1^2k_2^2}. \tag{2.51}$$

This procedure can be repeated until all descending powers of π have been exhausted, hence a complete closed form for the integrals $c(n,p)$ will be made possible.

Theorem 2.11. Let $n,\,p \in \mathbb{N}$ and let $\xi_p = \lfloor \frac{p}{2} \rfloor$. Then the even branches $X_n(p) = c(2n,p)$ of the integral

$$c(n,p) = \int_0^{\pi/2} x^p \cos^n x \, dx \tag{2.52}$$

are given by

$$X_n(p) = \sum_{j=0}^{\xi_p} a_{n,p,p+1-2j} \pi^{p+1-2j} + \delta_{\text{odd},p} \cdot a^*_{n,p}, \tag{2.53}$$

and the value of $a_{n,p,p+1-2j}$ for $p \geq 2$ and $0 \leq j \leq \xi_p$ is given by

$$a_{n,p,p+1-2j} = \frac{(-1)^j \binom{2n}{n} p!}{2^{2n+p+1}\,(p+1-2j)!} \sum_{1 \leq k_1 \leq k_2 \leq \cdots \leq k_j \leq n} \frac{1}{k_1^2 k_2^2 \cdots k_j^2},$$

and

$$a^*_{n,p} = \frac{(-1)^{\xi_p} \binom{2n}{n} p!}{2^{2n}} \sum_{1 \leq k_1 \leq k_2 \leq \cdots \leq k_p \leq n} \frac{1}{k_1^2 k_2^2 \cdots k_p^2} \sum_{j=1}^{k_p} \frac{2^{2j}}{j^2 \binom{2j}{j}}. \tag{2.54}$$

Similarly, for the odd branches $Y(n,p) = c(2n+1,p)$ we have

$$Y_n(p) = \sum_{j=0}^{\xi_p} b_{n,p,p-2j} \pi^{p-2j} + \delta_{\text{odd},p} \cdot b^*_{n,p}, \tag{2.55}$$

with

$$b_{n,p,p-2j} = \frac{(-1)^j \, p! \, 2^{2n+2j-p}}{(2n+1)\binom{2n}{n}\,(p-2j)!} \sum_{0 \leq k_1 \leq k_2 \leq \cdots \leq k_j \leq n} \frac{1}{(2k_1+1)^2 (2k_2+1)^2 \cdots (2k_j+1)^2},$$

and

$$b^*_{n,p} = \frac{(-1)^{\xi_p} \, p! \, 2^{2n}}{(2n+1)\binom{2n}{n}} \sum_{0 \leq k_1 \leq k_2 \leq \cdots \leq k_p \leq n} \frac{1}{(2k_1+1)^2 (2k_2+1)^2 \cdots (2k_p+1)^2} \sum_{j=0}^{k_p} \frac{\binom{2j}{j}}{2^{2j}(2j+1)}.$$

3. Some examples on the halfline

In this section we provide an analytic expression for

$$C_n(p,b) = \int_0^\infty x^{-p} \cos^{2n+1}(x+b)\, dx, \tag{3.1}$$

and

$$S_n(p,b) = \int_0^\infty x^{-p} \sin^{2n+1}(x+b)\, dx. \tag{3.2}$$

In the table [4] the evaluation of the special case $p = \frac{1}{2}$ and $b = 0$:

$$\int_0^\infty \frac{\cos^{2n+1} x}{\sqrt{x}}\, dx = \frac{1}{2^{2n}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^{n} \binom{2n+1}{n+k+1} \frac{1}{\sqrt{2k+1}}, \tag{3.3}$$

and

$$\int_0^\infty \frac{\sin^{2n+1} x}{\sqrt{x}}\, dx = \frac{1}{2^{2n}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{n+k+1} \frac{1}{\sqrt{2k+1}}, \tag{3.4}$$

as 3.822.2 and 3.821.14.

Theorem 3.1. Let $0 < p < 1$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Then

$$\int_0^\infty x^{-p} \cos^{2n+1} x \, dx = \frac{\Gamma(1-p)}{2^{2n}} \sin\left(\frac{\pi p}{2}\right) \sum_{k=0}^{n} \frac{\binom{2n+1}{n-k}}{(2k+1)^{1-p}}, \tag{3.5}$$

and

$$\int_0^\infty x^{-p} \sin^{2n+1} x \, dx = \frac{\Gamma(1-p)}{2^{2n}} \cos\left(\frac{\pi p}{2}\right) \sum_{k=0}^{n} (-1)^k \frac{\binom{2n+1}{n-k}}{(2k+1)^{1-p}}. \tag{3.6}$$

Proof. The identity $2\cos x = e^{ix} + e^{-ix}$ and the binomial theorem yield

$$\int_0^\infty x^{-p} \cos^{2n+1} x \, dx = 2^{-2n-1} \sum_{k=0}^{n} \binom{2n+1}{k} \int_0^\infty x^{-p} \left(e^{i(2n+1-2k)x} + e^{-i(2n+1-2k)} \right) dx. \tag{3.7}$$

Recall the Heaviside step function defined by $H(x) = 1$, if $x > 0$ and $H(x) = 0$ otherwise. Then, each of the integrals in (3.7) is evaluated using the Fourier transform

$$\int_{-\infty}^\infty H(x) x^{-p} e^{-i\omega x} \, dx = \frac{\Gamma(1-p)}{|\omega|^{1-p}} \exp(-i\pi(1-p)\text{sign}(\omega)/2). \tag{3.8}$$

□

Corollary 3.2. Let $p > 1$ be real and $n \in \mathbb{N}_0$. Then

$$\int_0^\infty \cos^{2n+1} x^p \, dx = \frac{1}{2^{2n}} \Gamma\left(\frac{p+1}{p}\right) \cos\left(\frac{\pi}{2p}\right) \sum_{k=0}^{n} \frac{\binom{2n+1}{n-k}}{(2k+1)^{1/p}}, \tag{3.9}$$

and

$$\int_0^\infty \sin^{2n+1} x^p \, dx = \frac{1}{2^{2n}} \Gamma\left(\frac{p+1}{p}\right) \sin\left(\frac{\pi}{2p}\right) \sum_{k=0}^{n} (-1)^k \frac{\binom{2n+1}{n-k}}{(2k+1)^{1/p}}. \tag{3.10}$$

Proof. The change of variables $x \mapsto x^{1/(1-p)}$ in the results of Theorem 3.1 gives the result. □

The last result described here is a further generalization of Theorem 3.1.

Theorem 3.3. Assume $b \in \mathbb{R}$, $0 < p < 1$ and $n \in \mathbb{N}_0$. Define

$$C_n(p,b) = \int_0^\infty x^{-p} \cos^{2n+1}(x+b) \, dx \tag{3.11}$$

and

$$S_n(p,b) = \int_0^\infty x^{-p} \sin^{2n+1}(x+b) \, dx. \tag{3.12}$$

Then

$$C_n(p,b) = \frac{\Gamma(1-p)}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{n-k} \frac{\sin(\frac{\pi p}{2} - (2k+1)b)}{(2k+1)^{1-p}}, \tag{3.13}$$

and

$$S_n(p,b) = \frac{\Gamma(1-p)}{2^{2n}} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{n-k} \frac{\cos(\frac{\pi p}{2} - (2k+1)b)}{(2k+1)^{1-p}}. \tag{3.14}$$

Proof. Denote the left-hand side of (3.13) and (3.14) by $f_n(b)$ and $g_n(b)$ respectively. Differentiation with respect to the parameter b yields

$$\begin{aligned}\frac{\partial g_n}{\partial b} - (-1)^n(2n+1)f_n &= (2n+1)\sum_{j=0}^{n-1}(-1)^j\binom{n}{j}f_j(b) \\ \frac{\partial f_n}{\partial b} + (-1)^n(2n+1)g_n &= -(2n+1)\sum_{j=0}^{n-1}(-1)^j\binom{n}{j}g_j(b).\end{aligned} \tag{3.15}$$

Considering b and p fixed, we now show that the right-hand side of (3.13) and (3.14) satisfy the system (3.15) with the same initial conditions. This will establish the result.

In the case of the right-hand side of (3.13), it is required to check the identity

$$2^{-2n}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{n-k}\frac{\sin(\frac{\pi}{2}p-(2k+1)b)}{(2k+1)^{1-p}} =$$

$$(2n+1)\sum_{j=0}^{n}(-1)^j\binom{n}{j}2^{-2j}\sum_{k=0}^{j}\binom{2j+1}{j-k}\frac{\sin(\frac{\pi}{2}p-(2k+1)b)}{(2j+1)^{1-p}}.$$

To verify this we compare the coefficients of the transcendental terms

$$\frac{\sin(\frac{\pi}{2}p-(2k+1)b)}{(2k+1)^{1-p}}.$$

It turns out that this question is equivalent to validating the identity

$$(-1)^k2^{-2n}\binom{2n+1}{n-k}(2k+1) = (2n+1)\sum_{j=k}^{n}(-1)^j2^{-2j}\binom{n}{j}\binom{2j+1}{j-k} \tag{3.16}$$

To this end, we employ the WZ-technology as explained in [14]. This method produces the recurrence

$$2(n+k+1)(n+1-k)u(n+1-k)-(n+1)(2n+3)u(n,k)=0. \tag{3.17}$$

To prove (3.16) simply check that both sides of (3.16) satisfy the recurrence (3.17) as well as the initial condition $u(0,0)=1$.

The identities

$$\begin{aligned}\int_0^\infty x^{-p}\cos(x+b)\,dx &= -\Gamma(1-p)\sin(b-\tfrac{p\pi}{2}) \\ \int_0^\infty x^{-p}\sin(x+b)\,dx &= \Gamma(1-p)\cos(b-\tfrac{p\pi}{2}),\end{aligned} \tag{3.18}$$

which are special cases of

$$\begin{aligned}\int_0^\infty x^{-p}\cos(ax+b)\,dx &= -a^{p-1}\Gamma(1-p)\sin(b-\tfrac{p\pi}{2}) \\ \int_0^\infty x^{-p}\sin(ax+b)\,dx &= a^{p-1}\Gamma(1-p)\cos(b-\tfrac{p\pi}{2}),\end{aligned} \tag{3.19}$$

show that the corresponding initial values in (3.13) (respectively 3.14) match. The evaluations (3.19) appear as 3.764.1 and 3.764.2 in [4]. To establish (3.18) expand $\cos(x+b)$ as $\cos x\cos b-\sin x\sin b$, use the change of variables $x\mapsto x^p$, and Theorem 3.1. □

We now discuss some definite integrals that follow from Theorem 3.3.

Example 3.1. Differentiating (3.13) with respect to p and setting $p = \frac{1}{2}$ and $b = 0$ gives, after the change of variables $x \mapsto x^2$,

(3.20)
$$\begin{aligned} \int_0^\infty \log x \, \cos^{2n+1} x^2 \, dx &= -\frac{\sqrt{\pi}}{2^{2n+3}}(\pi + 2\gamma + 4\log 2) \sum_{k=0}^n \binom{2n+1}{n-k} \frac{1}{\sqrt{4k+2}} \\ &- \frac{\sqrt{\pi}}{2^{2n+2}} \sum_{k=0}^n \binom{2n+1}{n-k} \frac{\log(2k+1)}{\sqrt{4k+2}}, \end{aligned}$$

where we have used the value $\Gamma'(1/2) = -\sqrt{\pi}(\gamma + 2\log 2)$.

Example 3.2. Assume $0 < p,\, q < 1$. Multiplying (3.13) by b^{-q} and integrating over the half-line yields (after replacing b by y)

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{\cos^{2n+1}(x+y)}{x^p y^q} dA &= -\Gamma(1-p)\Gamma(1-q)\cos\left(\frac{\pi(p+q)}{2}\right) \\ &\times \sum_{k=0}^n \binom{2n+1}{n-k} \frac{(2k+1)^{p+q-2}}{2^{2n}}. \end{aligned}$$

In particular, for $n = 0$,

$$\int_0^\infty \int_0^\infty \frac{\cos(x+y)}{x^p y^q} dA = -\Gamma(1-p)\Gamma(1-q)\cos\left(\frac{\pi(p+q)}{2}\right). \tag{3.21}$$

The derivative $\frac{\partial^2}{\partial p \partial q}$ at $p = q = \frac{1}{2}$ produces the evaluation

$$\int_0^\infty \int_0^\infty \frac{\log x \, \log y}{\sqrt{xy}} \cos(x+y) \, dx \, dy = (\gamma + 2\log 2)\pi^2 \tag{3.22}$$

that we promised in the Introduction.

Example 3.3. Iterating the method described in the previous example yields

$$\int_{\mathbb{R}_+^n} (\cos \|x\|^2) \cdot \prod_{j=1}^n \log x_j \, dV = \frac{(-1)^{\Delta_n} \pi^{n/2}}{2^{2n}} \begin{cases} \text{Re}\, \psi_n & \text{if } n \text{ is even,} \\ \text{Im}\, \psi_n & \text{if } n \text{ is odd,} \end{cases}$$

with

$$\Delta_n = \frac{n(n+1)}{2}, \; \psi_n = \left(\gamma + 2\log 2 + \frac{\pi i}{2}\right)^n e^{\pi i n/4}. \tag{3.23}$$

Here $\|x\|^2 = x_1^2 + \cdots + x_n^2$ and γ is Euler's constant. For instance, for $n = 3$ we have

$$\int_0^\infty \int_0^\infty \int_0^\infty \log x \log y \log z \cos(x^2+y^2+z^2) \, dx \, dy \, dz = \frac{\pi^{3/2}}{8}(-16\xi^3 + 12\xi^2\pi + 6\xi\pi^2 - \pi^3),$$

where $\xi = \gamma + 2\log 2$.

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